

Measurement of the frequency dependence of the ultrasonic parametric threshold amplitude for a fluid-filled cavity

A. Teklu

Department of Physics & Astronomy, College of Charleston, 66 George Street, Charleston, South Carolina 29424

Michael S. McPherson, M. A. Breazeale, Roger D. Hasse, and Nico F. Declercq^{a)}

National Center for Physical Acoustics, The University of Mississippi, 1 Coliseum Drive, University, Mississippi 38677

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By driving a transducer at one end of a fluid-filled cavity parallel to a rigid plane reflector at the other end, standing ultrasonic waves can be generated. Variations in the cavity length resulting from transducer motion lead to the generation of resonant frequencies lower than the drive frequency (known as fractional harmonics). This excitation of fractional harmonics in a liquid-filled cavity by ultrasonic waves was described previously as a parametric phenomenon [Laszlo Adler and M. A. Breazeale, *J. Acoust. Soc. Am.* **48**, 1077–1083 (1970)]. This system was modeled by using a modified Mathieu's equation whose solution resulted in the prediction of critical threshold drive amplitude for the excitation of parametric oscillation. The apparatus used by Adler and Breazeale was recently refined for accurate measurements of the threshold amplitude for parametric excitation at frequencies ranging from 2 to 7 MHz. The measurements showed that in this range the threshold amplitude increases with increasing drive frequency in apparent discrepancy with the results of Adler and Breazeale. Analysis of the theory indicates, however, that both past and current results lie in two different stability zones and each is in agreement with the existing theory.

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I. BACKGROUND

Parametric excitation of a resonant system is the self-excitation of a non-characteristic mode of the system. This is caused by a periodic variation of a parameter of the system upon which the natural frequency of the mode depends.¹ For this excitation to occur, the resonant frequency of the system must be nearly equal to one-half the frequency at which the parameter varies. In addition, the parameter must vary with an amplitude that is greater than a critical threshold value. This threshold amplitude for the generation of fractional harmonics is determined by the amount of dissipation in the system as well as the relationship between the exciting frequency and the excited frequency. Stabilization of the amplitude depends on the amount of dissipation and nonlinearity of the system.

Numerous examples of parametric phenomena are observed in different branches of physics. Perhaps the most familiar example is the operation of a playground swing. The child on the swing must periodically vary the vertical position of the center of mass so that the effective length of the pendulum changes. Synchronization of this change with the oscillation causes the oscillation amplitude to increase gradually from a small initial amplitude. The important point is that the resonant frequency of the swing, determined by the

swing's effective length, is modulated by the periodic raising and lowering of the center of mass.¹ The amplitude is limited by the nonlinearity of the system.¹

Other examples include the following: a driven tuning fork with one prong changing the length of a stretched string;² transverse vibration of a rod under the action of a longitudinal periodic force at twice the resonant frequency;³ parametrically excited fractional phonons in solids;⁴ electro-mechanical oscillator;⁵ and liquid-filled ultrasonic resonant system operated as an acoustical parametric generator.^{6–8}

II. THEORY

Adler and Breazeale made the first detailed investigation of a fluid-filled cavity caused to parametrically resonate by ultrasound.⁹ Further investigations were made by Yen.¹⁰ Such a cavity has standing-wave solutions satisfying the acoustic boundary conditions only for a series of specific frequencies known as the resonant frequencies. The values of these frequencies depend on the physical dimensions and geometrical shape of the cavity.

The apparatus used in the experiments of Adler and Breazeale utilized a water-filled, one-dimensional cavity terminated by two parallel planes at each end of the cavity. One end is a rigid plane reflector and the other end is a transducer driven by an external rf voltage of frequency 2ω . As the transducer at the end of the cavity oscillates back and forth, the length of the cavity changes, and this variation of the cavity length also causes the resonance frequency to vary. In addition, nonlinear effects of the medium in which the waves

^{a)}Permanent address: George W. Woodruff School of Mechanical Engineering, Georgia Institute of Technology, 801 Ferst Drive, Atlanta, GA 30332-0405 and Georgia Tech Lorraine, 2 rue Marconi, 57070 Metz, France. Electronic mail: nico.declercq@me.gatech.edu

propagate cause the generation of frequency doublets. These frequencies (ω_1 and ω_2) occur at approximately half the drive frequency (2ω) in such a manner that $\omega_1 + \omega_2 \approx 2\omega$.^{11,12}

Parametric excitation is possible only if the energy input to the system due to periodic variation of its resonant frequencies reaches a critical threshold value that is large enough to overcome the energy dissipated by the system. Therefore, the amplitude of the varying parameter must satisfy a threshold condition. In fact, it has been observed that frequencies other than the resonant frequencies can be generated by the nonlinearity of the medium when any of the acoustical parameters of the system are varied periodically.¹¹ Since some of these frequencies have values below the drive frequency, they are usually referred to as subharmonics or fractional harmonics.

Another important criterion for the observation of parametric oscillation is the aspect ratio, defined as the ratio of transverse to longitudinal dimensions of the cavity. The aspect ratio is related to the Fresnel number, defined as¹²

$$F = \frac{r^2}{\lambda L}, \quad (1)$$

where r is characteristic of the transverse size of the cavity (for example, the square root of the area of a plane radiator or reflector), λ is the wavelength of the ultrasonic wave, and L is the longitudinal length of the cavity in the direction of propagation of the wave. As a result of a transition from an initially stable to an unstable state, parametric resonance can be observed when the aspect ratio or the Fresnel number is large ($F \geq 1$). For the cavity used here $F=95$, satisfying the physical conditions for the observation of parametric excitation.

For the fluid-filled cavity resonator, the vibration of the transducer causes the length of the cavity, and hence the resonant frequency, to be periodic functions of time. The time-dependent cavity length can be expressed as¹¹

$$l(t) = l_0(1 + h \cos 2\omega t) \quad (2)$$

with

$$h = \frac{A}{l_0}, \quad (3)$$

where A is the vibration amplitude of the drive transducer, l_0 is the unperturbed length of the cavity, and 2ω is the frequency of the driver. Similarly, the time-dependent resonant frequency may be expressed as

$$\omega_n(t) = \omega_{n_0}(1 + h \cos 2\omega t), \quad (4)$$

where ω_{n_0} is the unperturbed characteristic frequency of the n th mode given by

$$\omega_{n_0} = \frac{n\pi c}{l_0} \quad (5)$$

and c is the speed of sound in the medium.

The wave equation governing the particle displacement $y(x,t)$ in an ultrasonic wave in a dissipative medium is given by¹¹

$$\frac{\partial^2 y(x,t)}{\partial t^2} = c^2 \frac{\partial^2 y(x,t)}{\partial x^2} + \frac{\alpha c^3}{\omega^2} \frac{\partial^3 y(x,t)}{\partial x^2 \partial t}, \quad (6)$$

where α is the acoustical absorption coefficient of the medium. Assuming a solution of the form

$$y(x,t) = g(t) \sin \frac{n\pi x}{l(t)} \quad (7)$$

and using Eq. (2), one can rewrite the dissipative wave equation and get

$$\frac{d^2 g'}{dz^2} + (\bar{a} - 2q \cos 2z)g' = 0, \quad (8)$$

where

$$g'(z) = g(z) \exp\left(\frac{-a\alpha c}{\omega} z\right), \quad (9)$$

$$\bar{a} = \left(a - \frac{a^2 \alpha^2 c^2}{\omega^2}\right), \quad (10)$$

$$z = \omega t, \quad q = ah, \quad \text{and} \quad a = \frac{\omega_n^2}{\omega^2}. \quad (11)$$

Equation (8) is a differential equation whose solution is given by a damped Mathieu's function satisfying the boundary conditions that the particle displacement y must vanish at $x=0$ and $x=l_0$. Depending on the value of a , the excited frequencies can be integral or fractional multiples of the natural resonance frequency. Furthermore, the solutions can be stable, unstable, or neutral.¹³

Since the fractional harmonics are generated from nonlinear interactions in the medium, it is necessary to consider the effects that such nonlinearities might have on the threshold drive amplitude. Our analysis, and that in Ref. 1, shows that the nonlinearity does not impact the threshold amplitude h . Thus, Eq. (8) is sufficient for the analysis of the threshold drive amplitude behavior.

In Eq. (11), the term ω_n represents the set of possible harmonics generated by the system. When these harmonics correspond to parametric resonance regions (where $a=1$, $a=4$, $a=9$, etc.¹³), one can observe parametric resonance. The solution to Eq. (8) leads to a condition on the drive amplitude h , for the onset of parametric amplification. This condition defines a region of instability as follows:

$$h\omega \geq \sqrt{(\omega_n - \omega_o)^2 + (a\alpha c)^2}, \quad (12)$$

where the equality represents the threshold condition for the onset of parametric amplification. There are several possible zones where parametric amplification occurs. In the case $\omega_n = \omega \approx \omega_o$, we have the condition $a=1$, so that the first term under the radical vanishes. In this case, with the harmonics near a physical resonance of the system, the threshold behavior is limited only by the dissipation in the system. Hence, for $a=1$, we have a threshold given by

$$h\omega > \alpha c, \quad (13)$$

which was the case observed in the initial experiments of Adler and Breazeale.⁸ However, for the other regions of

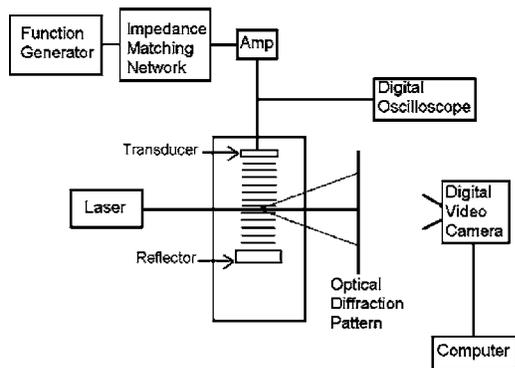


FIG. 1. Block diagram of the acoustic parametric generator.

parametric oscillation (with corresponding threshold amplitude minima at $a=4, 9, 16, 25$, etc.), one must include both terms under the radical. In the case to be described, $a=4$. This means that the threshold becomes

$$h\omega \geq \sqrt{(\omega_n - \omega_o)^2 + (4\alpha c)^2}. \quad (14)$$

In this case, the threshold condition is determined by both the attenuation and the resonant frequency.

Here, we report observation of the generation of fractional harmonics due to parametric oscillation in a liquid-filled, ultrasonic system similar to that originally used by Adler and Breazeale.⁸ The data show a more complicated threshold amplitude behavior than the $1/\omega$ frequency dependence observed by Adler and Breazeale⁸ (which is the behavior predicted by Eq. (13) for the $a=1$ region). The data in the present experiment require a fit with both terms of Eq. (12), and they correspond to the stability region defined by $a=4$. We have improved the oscillator stability and present new measurements of the frequency dependence of the drive threshold amplitude for the generation of parametric oscillation.

III. EXPERIMENT

A block diagram of the system used to study the frequency dependence of the threshold amplitude for generation of parametric resonances is shown in Fig. 1. It consists of a liquid-filled one-dimensional cavity bounded by a drive transducer (which acts as an oscillating boundary) at one end and by a rigid plane reflector at the other end. The reflector is a plane plate aligned parallel to the radiating transducer. The transducer is a 1-in. diameter, x-cut, quartz transducer driven by a rf signal generator. The impedance of the generator was matched to the quartz drive transducer by an impedance matching network. The frequencies ranged from 2 to 7 MHz.

The transducer-reflector assembly is designed to allow minute adjustments to achieve very accurate parallel alignment of the transducer and reflector. This assembly is immersed in a water-filled tank at room temperature. The periodically varying parameter in this case is the distance between the rigid reflector and the drive transducer, which changes as the transducer oscillates back and forth with a given drive frequency.

The periodic variations in the density of the liquid produced by a standing wave can act as an optical diffraction

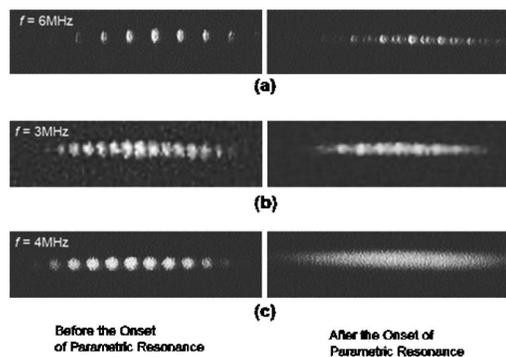


FIG. 2. Photos of diffraction patterns before and after the onset of parametric resonance: (a) The after photo shows a few intermediate orders; (b) The after photo shows a larger number of intermediate orders; and (c) The after photo shows essentially a continuum when the number of subharmonics becomes great enough.

grating. Laser light passing through the standing wave from a direction perpendicular to the cavity length diffracts and creates a diffraction pattern as shown on the left side in Fig. 2.

The presence of fractional harmonics produces extra diffraction orders between the ordinary diffraction orders resulting from standing waves in the cavity as shown on the right side of Fig. 2. Thus, the appearance of intermediate diffraction orders provides an indication of the onset of parametric excitation. The drive voltage was reduced until the extra diffraction orders disappeared in order to determine the minimum threshold voltage required for parametric excitation.

To observe parametric excitation, two experimental factors are critical: the parallelism between the transducer and reflector and the amplitude of the drive threshold. It is possible to overcome slight misalignment of the parallelism of the plate relative to the reflector by increasing the amplitude. Therefore, once parametric oscillation is observed, the parallelism is carefully adjusted so that the onset of parametric oscillation occurs at the lowest possible drive voltage.

IV. RESULTS AND DISCUSSIONS

Figure 2 shows diffraction patterns for three different frequencies both before and after the onset of parametric excitation. In Fig. 2(a), parametric resonance causes additional orders to appear. In Fig. 2(b), the number of intermediate orders has increased. In Fig. 2(c), the number of intermediate orders is so great that it is impossible to resolve individual orders in the spectrum, and they appear as a continuum.

Measurements of the threshold value of the transducer drive voltage for the onset of parametric oscillation were performed at different frequencies. Figure 3 shows a plot of the frequency dependence of the threshold drive voltage for the original data of Adler and Breazeale. As seen in the figure, the threshold value of the drive voltage decreases with increasing frequency. The theoretical curve indicated is a fit of Eq. (13) to the data. In the case of the original experiment of Adler and Breazeale, the physical parameters of the system preferentially excited the $a=1$ mode.

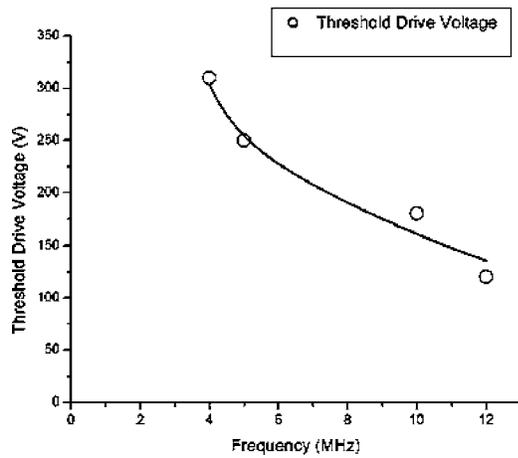


FIG. 3. Data of Adler and Breazeale (Ref. 8) with theoretical curve indicating stability criteria given in Eq. (13) (with $a=1$).

The data of the original experiment were fit with the theoretical expression given in Eq. (13). Dividing both sides of Eq. (13) by ω and using the equality to determine the minimum allowed value of the input parameter h , one has

$$h = \frac{1\alpha c}{\omega}. \quad (15)$$

Therefore, with $a=1$, the critical threshold amplitude has a $(1/\omega)$ dependence on drive frequency. This theoretical fit is shown in Fig. 3, and it is in good agreement with the data.

The results of the current experiment, which had different cavity dimensions, do not show a $(1/\omega)$ frequency dependence. However, by considering both terms in Eq. (14), one can understand why this is true. The region defined by $a=4$ also is a parametric region. Using $a=4$, one has a threshold dependence given in Eq. (14). This is the curve shown in Fig. 4 as a theoretical curve. This curve agrees well with the experimental data for a value of $a=4.08 \pm 0.72$, as shown, and indicates that the different frequency dependence of the threshold amplitude observed in this experiment results from the fact that the physical parameters of this experimental configuration (the same frequency range but considerably smaller cavity size than in the original experiment) are preferentially causing parametric excitation in the $a=4$ stability region.

V. CONCLUSIONS

The results of a new investigation of parametric resonance in a fluid-filled, one-dimensional cavity are presented. The threshold drive amplitude for the onset of parametric oscillations has been measured as a function of frequency. These data show an upward trend in apparent contradiction with previously published results.^{1,8} This apparent discrepancy

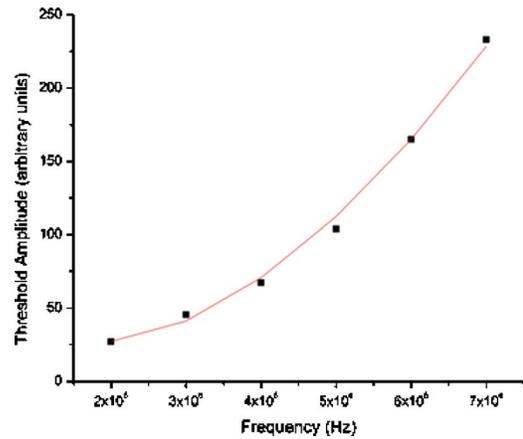


FIG. 4. Current results with theoretical curve indicating stability criteria given in Eq. (14) (with $a=4$).

is resolved by analysis of the theory. We demonstrate that the different experimental arrangements (with different characteristic frequencies) result in different regions of instability. This accounts for the different data trends.

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