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A useful analytical description of the coefficients in an inhomogeneous wave decomposition of a symmetrical bounded beam

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Abstract

If a bounded beam is described using a superposition of infinite inhomogeneous waves, the values of the coefficients attributed to each inhomogeneous wave are found using a classical optimization procedure, whence it is impossible to describe the obtained values analytically. In this paper, we develop a new and easy to apply straightforward analytical method to find the appropriate values of the sought coefficients. Supplementary to its analytical and straightforward nature, the method proves to reduce the inherent instabilities found in the inhomogeneous wave decomposition.

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1. Introduction

In order to describe a bounded beam, one can apply the classical Fourier decomposition [1]. This method describes the beam as a superposition of plane waves all traveling in a different direction and works extremely well in the bulk of a material. If, however, the interaction with an interface is to be described, the theory reveals some shortcomings. First, physical contradictions appear when narrow obliquely incident beams are described, because plane waves have to be considered with 'incidence angles' exceeding 90°. Second, it has been shown in numerous publications that critical phenomena such as Rayleigh wave generation cannot be described sufficiently accurate using pure plane incident waves. As a consequence, one is often compelled to apply the inhomogeneous wave decomposition of a bounded beam [2,3]. The latter describes a bounded beam as a superposition of inhomogeneous waves (i.e. having a complex wave vector), all traveling in the same direction, but having different inhomogeneities. Claeys and Leroy [2] were the first to apply this technique. They considered only positive inhomogeneities. Later, Van Den Abeele and Leroy [3] extended the theory and included also negative inhomogeneities. Their theory followed the same steps as taken by Claeys and Leroy [2] but was better in really tackling the generation of interface waves. Up until now, there has not been developed a straightforward tool to find the coefficients attributed to each inhomogeneous wave, nor has there been found any analytical expression to describe them. The method applied so far invokes a classical optimization procedure to find the 'best' value for each coefficient. It is well known that this method works extremely well inside the beam, but exponentially growing amplitudes appear beyond one or at the most four beam widths. In this paper, we introduce an analytical expression for the coefficients, therefore one is not obliged anymore to apply the more or less 'stochastic' tool of before. Furthermore, it will be shown by a

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numerical example that the obtained 'analytical curve' may not fit the gauss curve perfectly, but that the annoying exponential growth appears further away from the center of the beam than would be the case using the classical 'best fit' method [2–6]. This property can be very useful because interesting phenomena (such as the Schoch effect [7]) often appear beyond the point where the classical curve tends to be exponential. The analytical method introduced here may cast away this problem.

2. Theoretical development

A bounded beam $\varphi(x,z)$ with profile

$$\varphi(x,0)=f(x) \tag{1}$$

can be decomposed in a series of inhomogeneous waves [2–6]

$$\varphi(x,z)=\sum_{n=-N}^{N}\frac{1}{2}(1+\delta_{n,0})A_n\exp(\beta_n x)\times\exp\left(\mathrm{i}\sqrt{\left(\frac{\omega^2}{v^2}+\beta_n^2\right)}z\right) \tag{2}$$

where $\delta_{n,0}=1$ if $n=0$ and $\delta_{n,0}=0$ if $n\neq 0$, ω is the angular frequency, v is the velocity of plane sound waves, A_n is the amplitude that needs to be determined and β_n is the inhomogeneity attributed to the index n and is arbitrarily valued. Further below, in (5), we choose β_n to be proportional to n.

We solely reckon with symmetrical profiles, whence

$$\beta_n=-\beta_{-n}\quad\text{and}\quad A_n=A_{-n} \tag{3}$$

In signal processing and in ultrasonics, the Prony technique [8] has been used a few times before. It is based on the transformation of an equation, containing exponentials, into a polynomial, which unknown coefficients can be found using linear inversion procedures. The technique has been used before to describe any bounded beam profile as a superposition of Gaussian beams [9], but also, as in the current paper, to describe bounded beams as a superposition of infinite inhomogeneous waves [2,3]. In (2), the unknown coefficients A_n can be found using a specialized form of the Prony technique [2–6,8–10] by solving the equation

$$\sum_{n=0}^{N}A_n y^n=f(p\ln y) \tag{4}$$

with p being a constant value so that

$$y=\exp(x/p),\quad p=n/\beta_n \tag{5}$$

Eq. (4) can then be solved by applying a decomposition in Laguerre polynomials [2], whence

$$\sum_{n=0}^{N}B_nL_n(y)=f(p\ln y) \tag{6}$$

with

$$B_n=\frac{1}{p}\int_0^{+\infty}\exp(-y)f(p\ln y)L_n(y)\mathrm{d}y \tag{7}$$

and L_n the nth order Laguerre polynomial. Consequently, one demands [2,3]

$$A_n=\sum_{j=0}^{N}c_jB_j \tag{8}$$

It is not practical to find the unknowns c_j. The procedure is based on a least squares error estimation (or 'best fit' procedure) and the results are often tabulated for recycling purposes. Nevertheless, it might be useful to find an exact analytical expression for A_n. The latter has never been achieved thus far.We combine (4) and (5) and we utilize [11]

$$L_n(y)=\sum_{m=0}^{n}(-1)^m\frac{n!}{(n-m)!m!m!}y^m \tag{9}$$

whence

$$\sum_{n=0}^{N}A_ny^n=g \tag{10}$$

with

$$g=\sum_{n=0}^{N}\sum_{m=0}^{n}C_{m,n}y^m \tag{11}$$

$$C_{m,n}=B_n(-1)^m\frac{n!}{(n-m)!m!m!} \tag{12}$$

Now, we rewrite (11)

$$g=\sum_{q=0}^{N}D_qy^q \tag{13}$$

with

$$D_q=\sum_{n=q}^{N}C_{q,n} \tag{14}$$

Therefore coupling (10), (12)–(14), and requiring coefficients belonging to equal powers of y to be the same

$$A_n=\sum_{m=n}^{N}(-1)^n\frac{m!}{(m-n)!n!n!}\sum_{r=0}^{m}(-1)^r\frac{m!}{(m-r)!r!r!}I_r \tag{15}$$

with

$$I_r=\frac{1}{p}\int_{-\infty}^{+\infty}\exp\left[-\exp\left(\frac{x}{p}\right)+(r+1)\frac{x}{p}\right]f(x)\mathrm{d}x \tag{16}$$

Expression (15) is an analytical expression for finding the unknown coefficients A_n in (2).

It must be underlined that the values found by (15) may be different from those found using other techniques. That is because (15) is an analytical consequence of setting expression (4) equal to expression (5). The latter can only be analytically true for N infinitely large. Classical optimization procedures however demand (4) to be equal to (5) for a limited value of N, valid on a limited spatial interval. The concept is therefore different. The problem of the latter (least squares error estimation) approach is that the reconstructed profile differs a lot (exponentially!) beyond the limited spatial interval and when the interval is increased, the numerical optimization becomes numerically unstable and generates errors that again produce the 'exponential tails' at just a few (less than 4) beam widths from the center of the beam. The drawback of the analytical method presented here is that it is hard to have the numerical errors well in hand when solving expression (16), especially for large 'r'. Moreover, these errors cumulate dramatically in (15) and (2) if N is set large. These problems would vanish of course if the integral (16) could be solved analytically, which is as far as we know impossible.

Table 1
The values found for the coefficients $A_0, \ldots, A_7$ using a classical 'best fit method' and using the analytical method formulated in this paper

Coefficients	A classical best fit method	Analytical method
A_0	2009.790×10^5	-23727.757×10^{-4}
A_1	-1791.024×10^5	130729.573×10^{-4}
A_2	1264.645×10^5	-150052.951×10^{-4}
A_3	-702.482×10^5	66625.670×10^{-4}
A_4	302.940×10^5	-13804.836×10^{-4}
A_5	-99.244×10^5	1411.210×10^{-4}
A_6	23.847×10^5	-68.201×10^{-4}
A_7	-3.960×10^5	1.231×10^{-4}

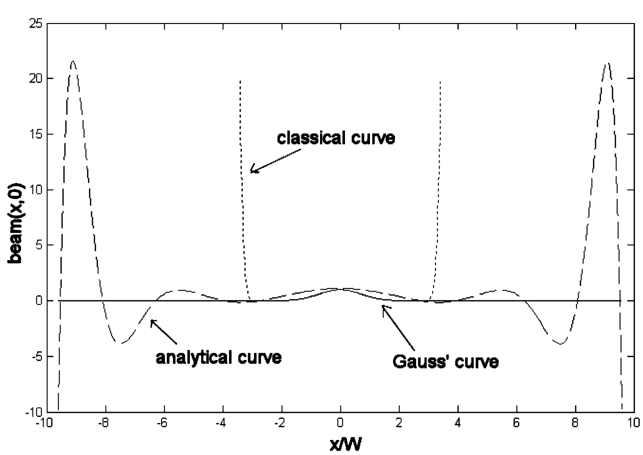

Fig. 1. Calculated curves for gaussian beam profiles. Solid line: the exact gaussian beam. Dotted line: approximation of the gaussian beam applying a classical optimization method. Dashed line: approximation of the gaussian beam applying analytical expression (15).

3. Numerical results

We have performed calculations for a Gaussian beam profile

$$f(x) = \exp(-x^2/W^2) \tag{17}$$

where W is the beam width and we have limited the number of inhomogeneous waves in (2) setting $N = 7$ and $p = 3.6W$. In Table 1, we have listed the obtained coefficients for the applied methods. The reconstructed profiles, invoking the coefficients of Table 1 are shown in Fig. 1, together with the perfect Gauss curve.

4. Conclusion

We have presented an analytical expression for the unknown coefficients A_n in an inhomogeneous wave decomposition of a bounded beam. Such an expression has never been reported elsewhere. The numerical problems that occur when implementing the analytical expression to determine the unknown coefficients A_n are less dramatic than the ones occurring in 'least squares error' methods used in the past.

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