ACTA ACUSTICA UNITED WITH ACUSTICA
Vol. 89 (2003) 1038 – 1040

The Principle of a Chopped Series Equilibrium to Determine the Expansion Coefficients in the Inhomogeneous Waves Decomposition of a Bounded Beam

Nico F. Declercq, Joris Degrieck
Soete Laboratory, Department of Mechanical Construction and Production, Faculty of Engineering, Ghent University, Sint Pietersnieuwstraat 41, 9000 Ghent, Belgium. NicoF.Declercq@rug.ac.be

Oswald Leroy
Interdisciplinary Research Center, KULeuven Campus Kortrijk, E. Sabbelaan 53, 8500 Kortrijk, Belgium

Summary
Up until now, the technique of expanding a bounded beam into inhomogeneous plane waves sometimes suffered from insurmountable numerical problems. Avoidance of these troubles was possible solely by throwing the expansion overboard and producing a new one with different parameters such as the spatial interval upon which optimization had to be performed, or the inhomogeneity interval that had to be taken under consideration. The latter often resulted in imperfect descriptions of critical phenomena such as beam displacement, for which inhomogeneous waves have proved again and again to be very well suited. Here, we introduce an ameliorated technique that determines the expansion coefficients by means of a chopped series representation of the inhomogeneous waves. We show that, if the series has an optimized length, the so found coefficients, which we attribute to the exact inhomogeneous waves in the expansion, are more accurate.

PACS no. 43.20.-f, 43.20.Bi, 43.20.El, 43.35.Pt

1. Introduction

It has been shown by many scientists that inhomogeneous waves are a great tool to describe the stimulation of critical waves like Rayleigh waves and Lamb waves [1, 2, 3, 4, 5, 6, 7, 8]. This fact formed the impetus to describe bounded beams in terms of inhomogeneous waves in order to deal with experimentally observed phenomena such as beam displacement [9, 10, 11, 12]. The latter description was first obtained by Claeys and Leroy [13] and was improved later by others [14, 15]. The inhomogeneous waves description of a bounded beam by inhomogeneous waves differs from the classical Fourier description [16] in that all inhomogeneous waves travel in the same direction but differ in inhomogeneity. However the most important reason why not everybody is excited to apply the inhomogeneous waves description of bounded beams, is the appearance of exponential tails beyond a certain distance away from the center of the bounded beam (at the most 4 beam widths, but often much less due to numerical problems during optimization), since certain phenomena that might be important often appear at distances beyond the appearance of the tails. These tails emerge as a consequence of numerical instabilities when optimization methods are applied. That is because exponential functions are extremely sensitive in the sense that very small optimization errors become visible as exponentially large errors. It may therefore be highly important to find a method that suppresses these optimization errors. This is the aim of this letter.

Received 20 August 2002,
accepted 20 June 2003.

2. Theoretical development

A narrow gaussian beam $\varphi(x,z)$ with profile $f(x)$ at $z = 0$ is to be described as a superposition of inhomogeneous waves

$$\varphi(x,0) = f(x) = \sum_{n=-N}^{N} A_n \exp(\beta_n x) \tag{1}$$

for $x \in [-L, L]$ in which β_n is called the inhomogeneity parameter, A_n is the corresponding amplitude and L determines the interval upon which the optimization occurs. If we apply a classical optimization method, then we find numbers A_n^* for the unknown coefficients A_n. Ideally, we would have $A_n^* = A_n$, but it is well known that on the contrary, $A_n^* \neq A_n$ whence

$$\sum_{n=-N}^{N} A_n^* \exp(\beta_n x) - f(x) = \xi(x) \neq 0. \tag{2}$$

Numerous calculations have convinced us that $\xi(x)$ depends on the range of β_n taken under consideration, on the number of inhomogeneous waves considered as well as on the interval on which the optimization is performed. Moreover, the behavior of the numerical error $\xi(x)$ as a function of these parameters is really unpredictable. The reason is not a matter of physics but purely a matter of the optimization being well or bad conditioned for the selected parameters. It is our aim to find a method that improves the conditioning of the optimization, whence numbers A_n^{**} for the unknown coefficients A_n are found, so that

$$\sum_{n=-N}^{N} A_n^{**} \exp(\beta_n x) - f(x) = \varepsilon(x), \tag{3}$$

with

$$|\varepsilon(x)| \leq |\xi(x)|. \tag{4}$$

First, we write $\exp(\beta_n x)$ as a series expansion so that

$$\exp(\beta_n x) = \sum_{r=0}^{\alpha} \frac{(\beta_n x)^r}{r!} + \Delta_n^\alpha(x), \tag{5}$$

with

$$\Delta_n^\alpha(x) = \sum_{r=\alpha+1}^{\infty} \frac{(\beta_n x)^r}{r!}. \tag{6}$$

We now seek for the coefficients A_n^{**} for which

$$\sum_{n=-N}^{N} A_n^{**} \left(\sum_{r=0}^{\alpha} \frac{(\beta_n x)^r}{r!} \right) = f(x) \tag{7}$$

for $x \in [-L, L]$. There will still be a numerical error on this calculation, so that

$$\sum_{n=-N}^{N} A_n^{**} \left(\sum_{r=0}^{\alpha} \frac{(\beta_n x)^r}{r!} \right) - f(x) = \mu(x). \tag{8}$$

It is known from our experience and from the simple fact that no polynomial is more difficult to deal with numerically in an optimization than an exponential function, that always

$$|\mu(x)| < |\xi(x)|, \tag{9}$$

and also that

$$\frac{\partial|\mu(x)|}{\partial|x|} < \frac{\partial|\xi(x)|}{\partial|x|}. \tag{10}$$

Now, if we utilize the found coefficients A_n^{**} in (1), we will obtain

$$\sum_{n=-N}^{N} A_n^{**} \exp(\beta_n x) - f(x) = \varepsilon(x) \neq 0. \tag{11}$$

The error $\varepsilon(x)$ contains an intrinsic contribution $\rho(x)$ due to the chop process and the numerical contribution $\mu(x)$ of (8), whence

$$\varepsilon(x) = \rho(x) + \mu(x). \tag{12}$$

Now, for α small in (8), $\mu(x) < \rho(x)$. As α ups, $\mu(x)$ increases while $\rho(x)$ decreases. Consequently, $\varepsilon(x)$ shows a minimum for a certain α such that $\mu(x) = \rho(x)$ and hence $\varepsilon(x) = 2\mu(x)$. We call this situation the chopped series equilibrium. This equilibrium error $\varepsilon(x)$ can be larger, equal to or smaller than $\xi(x)$.

Besides this, there is also the dependence of the numerical error on the interval on which the optimization process is performed and on the number of inhomogeneous waves concerned. It is clear from the fact that functions that tend to exponentials are hard to utilize in an optimization if the function argument becomes large, that the larger the interval upon which the optimization process is performed, the larger the numerical errors will be. In other words, as L increases, the optimization upon the interval $[-L, L]$ shows larger numerical errors. As a consequence, $\mu(x)$ increases with L, but less significantly than $\xi(x)$. Hence, it is possible to obtain a situation in which $2\mu(x) < \xi(x)$ and hence $\varepsilon(x) = 2\mu(x)$ will be less than $\xi(x)$. If that situation does or does not occur depends on all parameters involved.

Hence we have shown that situations occur for which there exists an α for which $\varepsilon(x)$ shows a minimum that is smaller than $\xi(x)$. We have therefore found a method that in certain circumstances enhances the optimization process to find the expansion coefficients in an inhomogeneous waves decomposition of a bounded beam; hence the exponential tails that appear will be shifted outwards.

3. Numerical results

We have performed a numerical optimization in order to find the expansion coefficients in (1) for a gaussian profile

$$f(x) = \exp\left(-x^2/W^2 \right). \tag{13}$$

As in [13, 14, 15], we have taken

$$\beta_n = n/p, \tag{14}$$

with p a real number. We have plotted the exact profile (13) as a function of x/W in a dotted line and the obtained curve for the description of that profile by means of inhomogeneous waves in a solid line. In Figures 1 and 2, the upper plot is the case where we have worked directly with the exponential functions, while the lower plot is the case by means of the chopped series equilibrium' introduced here. In both cases, $N = 29$, $L = 6.5W$, $a = 100$. In Figure 1, $p = 3W$. We notice that in this case the optimization using the exponentials to obtain A_n in (1) is quite stable and so is the optimization using the chopped series expansion. In Figure 2, we have taken $p = 6.7W$, for which the optimization using exponentials becomes very unstable, while we see that the optimization using the chopped

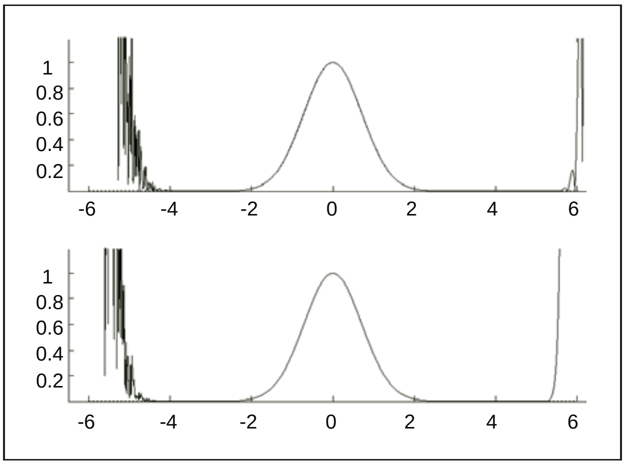

Figure 1. Results for $p = 3W$. Dotted line: exact gaussian beam profile as a function of x/W. Solid line: inhomogeneous waves decomposition. Top: old method using A_n^*, see (2). Bottom: new method using A_n^{**}, see (3).

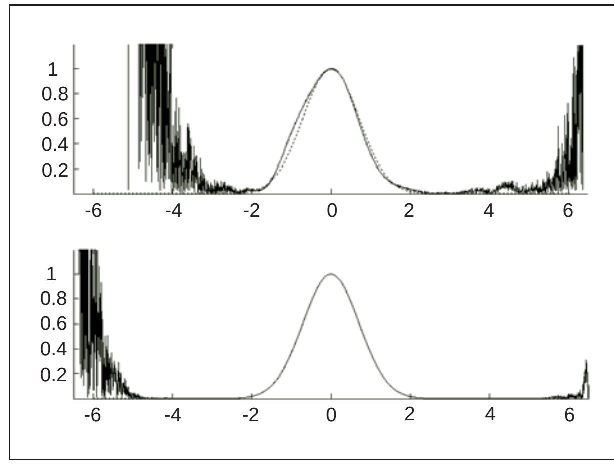

Figure 2. Results for $p = 6.7W$. Dotted line: exact gaussian beam profile as a function of x/W. Solid line: inhomogeneous waves decomposition. Top: old method, method using A_n^*, see (2). Bottom: new method using A_n^{**}, see (3).

series expansion is far more stable. In the case of Figure 2, we conclude that the technique that is introduced here gives much better results.

4. Conclusion

A technique is found that diminishes numerical instabilities in the determination of the expansion coefficients of an inhomogeneous waves decomposition of bounded beams. This result is important because up until now the inhomogeneous waves decomposition of bounded beams often suffered from numerical instabilities that could only be overcome by altering the parameters in the decomposition, whence these became insufficiently optimized to describe critical phenomena such as beam displacement.

Acknowledgement

The authors wish to thank 'The Flemish Institute for the Encouragement of the Scientific and Technological Research in Industry (I.W.T.)' for sponsoring.

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